Rotator Model of Elementary Particles Considered as Relativistic Extended Structures in Minkowski Space*

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The purpose of this paper is to investigate some consequences of the assumption that elementary particles are not pointlike, but are rather, extended structures in Minkowski space.

In terms of the hypothesis that the internal quantum states of such structures correspond to internal "rotator" levels belonging to the Hilbert space containing all irreducible finite-dimensional representations of the group SO_3^* of three-dimensional complex rotations (isomorphic to the Lorentz group), we obtain a particle classification which recovers (including leptons) the Nishijima–Gell-Mann classification of elementary particles. In this way, we justify the empirical Nishijima–Gell-Mann relation between isobaric spin, strangeness, baryon number, and charge. Moreover, as will be shown in a second paper, the new internal ("hidden") degrees of freedom which correspond to isobaric spin, strangeness, and baryon number open up new possibilities for understanding qualitatively and quantitatively the elementary particle interactions and decays; while a simple extension of "fusion" theory yields possible external state vectors and equations associated with any given internal quantized states corresponding to known elementary particles.

INTRODUCTION

IN a series of preceding papers[1] the idea, proposed by two of us (DB and JPV), of dropping the point-particle model and of introducing new kinematical variables in Minkowski space-time in order to represent extended material distributions enclosed within timelike tubes, has been systematically developed. Briefly, the reasons for doing so are the following:

(a) Recent high-energy collision experiments by Hofstadter and his collaborators[2] suggest that particles are not points moving along timelike world lines, as assumed usually in quantum theory, but are instead material distributions extended in space.

(b) The new quantum numbers, isobaric spin, strangeness, and baryon number, should be associated to new degrees of freedom, that is, to new "internal" collective kinematical variables. Now the usual point of view is that such variables, if they exist, belong to a new abstract space (assumed, in general, to be 3 or 4 dimensional with Euclidean metric) quite independent of Minkowski space-time. As a result, "external" and "internal" motions are absolutely disconnected and there is no relation between them. This at first sight does not seem satisfactory and it is tempting to investigate the possibility that these "internal" variables correspond in fact to new "hidden" over-all internal motions of extended structures endowed with definite symmetries in physical space-time. In this case, one expects that there should appear a connection between external space-time and internal isobaric spin space; and we find, indeed, that external and internal states belong to finite-dimensional representations of isomorphic groups acting, as we shall see, on different manifolds.[3] (This was also assumed independently by Iwanenko.[4])

(c) Part of the divergence difficulties of present quantum field theory are bound up with the assumed pointlike aspect of particles and their representation by δ functions. The introduction of extended models as starting point is reasonable since they offer the possibility of a natural "cutoff" if their dimensions are $\simeq$ to 0.6×10^{-13} cm.

* This paper is a summary of the results of a common program of research on a particularly simple rotator model started three years ago in the Institut Henri Poincaré in collaboration with David Bohm. Since then, starting from the same basic idea, the authors have investigated a wide range of more complex models which will be discussed and interpreted in subsequent papers. The contribution of each author is indicated as far as possible in the text itself. Notations are as usual: All Latin indexes vary from one to three, Greek indexes vary from one to four (with $x_4=ict$); repeated indexes implying the classical summation convention.

[1] D. Bohm and J. P. Vigier, Phys. Rev. **109**, 1882 (1958); F. Halbwachs, P. Hillion, and J. P. Vigier, Nuovo Cimento **15**, 209 (1960).

[2] R. Hofstadter, Revs. Mod. Phys. **28**, 214 (1958).

[3] The external Lorentz group acts on Minkowski space while the internal group operates on the manifold of the three-dimensional complex rotation group.

[4] A. Brodski and D. Ivanenko, Nucl. Phys. **13**, 447 (1959).

(d) Finally, several of the authors have further reasons[5] for assuming that particles are indeed extended, since this notion is part of the scheme which they have proposed in the frame of the causal interpretation of quantum mechanics (in particular, the idea of the possibility of hidden variables).

We shall attempt to develop the idea of extended particles in the following way.

In the absence of any clear experimental indication concerning the new degrees of freedom, we start from the indirect hint given by the Nishijima–Gell-Mann scheme of internal quantum number and try to discover a new "internal" group G which corresponds to possible internal invariance properties *compatible with real space-time structures*. The finite-dimensional irreducible representations of G should then yield the desired quantum numbers and suggest possible physical interpretations of the new degrees of freedom.

More precisely, we consider the original extended model as an heuristic analogy which serves only to help us to discover the new internal mathematical structure. One proceeds at some stage to pass from the assumed—but as yet unknown—extended structure to an abstract model defined by its invariance under a suitable internal group (our new isobaric group G), maintaining, however, general requirements for a structure in Minkowski's space-time. This line of research was essentially proposed by Finkelstein in a very interesting paper[6] which anticipates some of our results.

I. NONRELATIVISTIC MODEL

We start from the basic assumption that the isolated particle, described in first approximation by the ordinary pointlike picture, can be treated, in a second approximation, in terms of two kinematical frames (orthonormal relativistic tetrads) $a_\mu^{(\xi)}(\tau)$ and $b_\mu^{(\xi)}(\tau)$ centered on the same point $x_\mu(\tau)$, which latter coincides with the above pointlike picture. The parameter τ is the proper time along the world line followed by x_μ. The index ξ ($\xi=1, 2, 3, 4$) labels the vectors and μ ($\mu=1, 2, 3, 4$) their components; $\xi=4$ corresponding to a timelike vector and $\mu=4$ to the time components. $a_\mu^{(\xi)}$ and $b_\mu^{(\xi)}$ will be called for short the L and T frames.

This model may be called a *relativistic rotator.* Its fundamental character is the localization, not only of the center x_μ, but also of the two sets of vectors, in the frame of Minkowski space-time, so that the supplementary "internal" parameters needed for the description of the various particles have a "realistic" meaning related to the relativistic world (unlike what happens in the usual theories implying some abstract "isobaric spin spaces"). Such a model is able to lead to many kinds of extended pictures, but all the latter ones have the common character that they are describable as structures in Minkowski space-time. This is indeed the basic characteristic the present theory.

The complete treatment of the so-defined rotator needs two kinds of parameters: first, the "external" parameters which characterize the so-called "fixed," or L, tetrad, with respect to an arbitrary relativistic "laboratory" frame, namely, the relativistic coordinates x_μ of the rotator center and the parameters defining the orientation of the tetrad (we have thus ten parameters); second, the six "internal" parameters which define the relative orientation of the T tetrad with respect to the L tetrad.

Now in addition, we shall make two supplementary assumptions which simplify the problem.

First, the laws of evolution of the internal parameters, that is, the laws of the relative motion of both tetrads, are independent of the external parameters (of the global motion of the structure considered as a block) at least in the absence of interaction with external fields. Thus, the six internal kinematical parameters have an intrinsic character and define, as we shall see, the manifold on which operates our new isobaric spin group G. The two first sections of this paper are devoted to the separate study of this internal motion.

Second, the internal structure is assumed to possess spherical symmetry; that is, the T frame, considered as a simplified description of the structure, can be chosen arbitrarily, at least as for its spacelike vectors, in the same way that for the description of the classical motion of a sphere all the systems of orthogonal axes rigidly bound to the sphere are equivalent.

As a first step, let us consider the *nonrelativistic limit* and assume that the particle structure is represented by two kinematical three-dimensional frames $a_k^{(r)}$, $b_k^{(r)}$, the internal configurations corresponding to the relative orientation of $b_k^{(r)}$ with respect to $a_k^{(r)}$ expressed by three independent parameters. This means the particle geometry is the form of the material distribution carried along by the parameter transformations through which we denote the chosen internal motion. Whatever shall be the chosen parameters, it is clear that the internal state of the nonrelativistic model is represented by a definite element of the rotation group SO_3.

We now recall well-known considerations about the rotation group in order to particularize the manifold of the group.

For the unit vector and the rotation axis having the components:

$$Y_1=\sin\beta\cos\alpha,\quad Y_2=\sin\beta\sin\alpha,\quad Y_3=\cos\beta \tag{1}$$

(where α and β are its spherical coordinates), one takes the point with measure $\sin\gamma$ on this vector (2γ is the

[5] See L. de Broglie, *Une tentative d'interprétation causale . . .* (Gauthier-Villars, Paris, 1956); D. Bohm, Phys. Rev. **85**, 166 and 180 (1952); D. Bohm and J. P. Vigier, *ibid.* **96**, 208 (1954); J. P. Vigier, *Structure des microobjets* (Gauthier-Villars, Paris, 1957).

[6] D. Finkelstein, Phys. Rev. **100**, 924 (1955).

rotation angle) and one defines the point P:

$$y_1=\sin\gamma\,\sin\beta\,\cos\alpha,\quad y_2=\sin\gamma\,\sin\beta\,\sin\alpha,\quad y_3=\sin\gamma\,\cos\beta. \tag{2}$$

Evidently, there is a bicontinuous correspondence between the points P which fill the bowl with radius 1, and the elements of the rotation group SO_3. But this correspondence is neither one-to-one in the one sense nor in the other, since the different rotations γ and $\pi-\gamma$ around the same axis correspond to the same point P. On the other hand, the points (y_1, y_2, y_3) and $(-y_1, -y_2, -y_3)$ correspond to opposite rotations around opposite axis, which are identical. The first ambiguity is removed by considering the space R_3 to be inside a four-dimensional Euclidean space R_4, and by endowing the point P with a fourth coordinate, $\cos\gamma$:

$$y_1=\sin\gamma\,\sin\beta\,\cos\alpha,\quad y_2=\sin\gamma\,\sin\beta\,\sin\alpha,\quad \times y_3=\sin\gamma\,\cos\beta,\quad y_0=\cos\gamma. \tag{3}$$

As a consequence, P now lies on the hypersphere S_3 with radius 1 in the four-dimensional space R_4. This hypersphere is the Riemannian manifold of the rotation group and the configuration space of the internal theory, provided two opposite points are regarded as identical (the hypersphere is the *covering manifold* of SO_3). In other terms, to each point y_μ of S_3 (all the y_μ are real and we have $y_\mu y_\mu=1$), corresponds a 3×3 rotation matrix $\Omega_y=\Omega_{-y}$ which carries $a_k^{(r)}$ on $b_k^{(r)}$:

$$b_k^{(r)}=\Omega_y^{rs}a_k^{(r)},\quad \Omega_y^{rs}\Omega_y^{rt}=\delta^{st},$$

or in matrix form

$$(b_k)=\Omega_y(a_k),\quad \Omega_y\Omega_y^T=1. \tag{4}$$

The internal motion of the particle is thus represented as the motion of a point on this hypersphere in the same way as the "external" motion of a pointlike particle is represented as the motion of a point in a Euclidean space R_3.

The theory is invariant *under the rotations independent of time acting on the fixed triad*, owing to the general rotation invariance of the nonrelativistic physics, and on the other hand *under the rotations independent of time acting on the moving triad*, owing to the arbitrary choice of the latter implied by the assumed spherical symmetry. With respect to the group SO_3, these transformations are, respectively, the right and the left translations on the group. More precisely, let us designate by Ω_y the rotation from fixed to moving frame, related to the moving point $y(t)$ on R_3, and by Ω_{y1} the rotation of the (a_μ) frame, related to the point y_1, and by Ω_{y2} the rotation of the (b_μ) frame, related to the point y_2. We pass from the frames (a_μ), (b_μ), to the transformed frames $(a_\mu')=\Omega_{y1}\cdot(a_\mu)$, $(b_\mu')=\Omega_{y2}\cdot(b_\mu)$. We then have $(b_\mu')=\Omega_{y'}\cdot(a_\mu')'$, with the new rotation $\Omega_{y'}$ from new fixed to new moving frame, related to the new moving point $y'(t)$ defined by the fundamental formula:

$$\Omega_{y'}=\Omega_{y2}\cdot\Omega_y\cdot\Omega_{y1}^{-1}. \tag{5}$$

Such a transformation acting on the elements of a group is referred to the following general conception due to J. M. Souriau. If G is some group with elements A, B, $C\cdots$, we call *bilateral transformation on* G that operation which transforms each element C into

$$C'=F_B{}^A(C)=ACB^{-1}. \tag{6}$$

Each transformation $F_B{}^A$ is thus labeled by two definite elements A, B of G. It is easily shown that all these transformations form a group, called the *bilateral group on* G: Bil(G), with the rules

$$F_B{}^A-F_{B'}{}^{A'}=F_{BB'}{}^{AA'},\quad (F_B{}^A)^{-1}=F_{B^{-1}}{}^{A^{-1}}. \tag{7}$$

Of course, two transformations $F_B{}^A$, $F_{B'}{}^{A'}$ associated with two different pairs of elements of G are not necessarily different. More precisely, let us consider the direct product $G^2=G\times G$ composed with the elements $\begin{pmatrix}A\\B\end{pmatrix}$ with the rules

$$\begin{pmatrix}A\\B\end{pmatrix}\begin{pmatrix}A'\\B'\end{pmatrix}=\begin{pmatrix}AA'\\BB'\end{pmatrix},\quad \begin{pmatrix}A\\B\end{pmatrix}^{-1}=\begin{pmatrix}A^{-1}\\B^{-1}\end{pmatrix}, \tag{8}$$

the unity being, of course, $\begin{pmatrix}1\\1\end{pmatrix}$. We can obviously associate with each element $\begin{pmatrix}A\\B\end{pmatrix}$ of G^2 the element $F_B{}^A$ of Bil(G) with the same multiplication rule, but in general this correspondence is not an isomorphism. Indeed, the unity of Bil(G) is defined by

$$I_B{}^A(C)=ACB^{-1}=C \tag{9}$$

for any C. In particular, for $C=1$ we have $AB^{-1}=1$, so that A and B are identical. Now the condition

$$ACB^{-1}=ACA^{-1}=C \tag{10}$$

for any C means that A commutes with all the elements of G, and belongs to the center $\mathfrak{C}$ of G. Thus all elements of Bil(G) expressed by $F_A{}^A$, A being any element of the center $\mathfrak{C}$, are identical to the unity of Bil(G); this unity corresponding to the different elements $\begin{pmatrix}A\\A\end{pmatrix}$ of G^2. The set $\begin{pmatrix}A\\A\end{pmatrix}$ with $A\in\mathfrak{C}$ builds the nucleus of the homomorphism between G^2 and Bil(G). In other words, *the bilateral group is isomorphic to the quotient group* $G^2/\mathfrak{C}$. In the case of the group SO_3 we have simply

$$\mathrm{Bil}(SO_3)=SO_3\times SO_3=SO_4/\mathcal{G} \tag{11}$$

($\mathcal{G}$ is the two-element group $1, -1$), so that our configuration hypersphere admits as invariance group the whole four-dimensional rotation group in R_4.

The algebra of our internal group G is easily established. The infinitesimal left and right translations can be expressed, respectively, by

$$y_\mu \to (1+\alpha^i L_i)_{\mu\nu} y_\nu, \quad y_\mu \to (1+\alpha'^i R_i)_{\mu\nu} y_\nu, \tag{12}$$

where α^i and α'^i are arbitrary real infinitesimal parameters ($i=1, 2, 3$), L_i and R_i are six 4×4 matrices which are the infinitesimal generators of the left and right translations on SO_3, respectively. L_i and R_i are indeed left and right quaternion bases as we have:

$$L_1 L_1 = R_1 R_1 = -1, \tag{13}$$

and the same holds for the other indices. Also

$$L_1^T = L_1^{-1} = -L_1, \tag{14}$$

and

$$L_1 L_2 = -L_2 L_1 = L_3, \tag{15}$$

with cyclic permutation. The same relations are valid for the R_i.

Finally all the L_i commute with all the R_k. We have then:

$$\delta y_\mu = (\alpha^i L_i + \alpha'^i R_i)_{\mu\nu} y_\nu. \tag{16}$$

Let now $F(y)$ be any function defined in R_4; we have, by putting $\partial_\mu = \partial/\partial y_\mu$:

$$\partial_i = \partial_\mu (L_i)_{\mu\nu} y_\nu, \quad \partial_i' = \partial_\mu (R_i)_{\mu\nu} y_\nu, \tag{17}$$

$$\delta F(y) = (\alpha^i \partial_i + \alpha'^i \partial_i') F(y). \tag{18}$$

The operators ∂_i and ∂_i' obey the relations:

$$\partial_i \partial_i = \partial_i' \partial_i' = \partial^2, \tag{19}$$

$$\begin{aligned} &[\partial_i, \partial_j] = -2\epsilon_{ijk}\partial_k, \quad [\partial_i', \partial_j'] = -2\epsilon_{ijk}\partial_k', \\ &[\partial_i, \partial_k'] = 0, \quad [\partial_i, \partial^2] = [\partial_i', \partial^2] = 0. \end{aligned} \tag{20}$$

According to the ordinary procedure, we deduce from these infinitesimal generators of the invariance group acting on the configuration manifold, the corresponding (Hermitian) quantum operators by multiplication with $\hbar/i$, namely,

$$J_k = (\hbar/i)\partial_k, \quad J_k' = (\hbar/i)\partial_k', \quad J^2 = -\hbar^2 \partial^2, \tag{21}$$

with the commutations rules

$$\begin{aligned} &[J_i, J_j] = (\hbar/i)\epsilon_{ijk} J_k, \quad [J_i', J_0'] = (\hbar/i)\epsilon_{ijk} J_k', \\ &[J_i, J_j'] = [J_i, J^2] = [J_i', J^2] = 0. \end{aligned} \tag{22}$$

These commutation relations imply evidently the existence of three commuting operators, such as, for example, J^2, J_3, J_3', and a corresponding series of corresponding eigenfunctions $Y(l; m, m')$ (generalized spherical functions) satisfying

$$\begin{aligned} &J^2 Y(l; m,m')(y) = \hbar^2 l(l+1) Y(l; m,m')(y), \\ &J_3 Y(l; m,m')(y) = \hbar m Y(l; m,m')(y), \\ &J_3' Y(l; m,m')(y) = \hbar m' Y(l; m,m')(y), \end{aligned} \tag{23}$$

where l can take all possible values $0, \frac{1}{2}, 1, \cdots$ while m and m' take independently all values $-l, -l+1 \cdots l-1, l$.

As one knows, any function F of y can be developed on the set of the functions $Y(l; m,m')(y)$. These functions span the whole Hilbert space of the functions of y. As Wigner has shown, the Y with fixed values for l and m' span an invariant subspace which transforms under the irreducible representation $D(l)$ of SO_3.

II. RELATIVISTIC MODEL

According to our program we now pass to the relativistic theory of extended particles. In the frame of the first line of research let us assume that extended classical relativistic particles can be represented schematically by two kinematical frames $a_\mu^{(\xi)}$ and $b_\mu^{(\xi)}$ centered on a moving kinematical point $x_\mu(\tau)$. These new internal variables form two frames called, respectively, L and T frames. Their relative orientations which correspond to the elements of the homogeneous Lorentz group, will define the configurations of our system.

As was shown by Cartan and by Einstein and Mayer, if one introduces the following set of skew self-dual and antidual tensors associated to $a_\mu^{(\xi)}$ and $b_\mu^{(\xi)}$ by

$$\begin{aligned} &\mathfrak{A}_{\mu\nu}^{(r)\pm} = \epsilon^{rst} a_\mu^{(s)} a_\nu^{(t)} \pm (a_\mu^{(r)} a_\nu^{(4)} - a_\nu^{(r)} a_\mu^{(4)}), \\ &\mathfrak{B}_{\mu\nu}^{(r)\pm} = \epsilon^{rst} b_\mu^{(s)} b_\nu^{(t)} \pm (b_\mu^{(r)} b_\nu^{(4)} - b_\nu^{(r)} b_\nu^{(4)}), \end{aligned} \tag{24}$$

one can form their three independent components:

$$\begin{aligned} &A_k^{(r)\pm} = a_k^{(r)} a_4^{(4)} - a_4^{(r)} a_k^{(4)} \pm \epsilon_{ijk} a_i^{(r)} a_j^{(4)}, \\ &B_k^{(r)\pm} = b_k^{(r)} b_4^{(4)} - b_4^{(r)} b_k^{(4)} \pm \epsilon_{ijk} b_i^{(r)} b_j^{(4)}. \end{aligned} \tag{25}$$

The $A_k^{(r)+}$ constitute a triad of complex orthonormal fixed vectors and the $B_k^{(r)+}$ a triad of complex orthonormal moving vectors, which span a three-dimensional complex Euclidean space E_3^+. In the same way $A_k^{(r)-}$ and $B_k^{(r)-}$ span a complex conjugate three-dimensional Euclidean space E_3^-. Now, if the relativistic tetrad $a_\mu^{(\xi)}$ undergoes any special Lorentz transform Λ, which carries it onto the tetrad $b_\mu^{(\xi)}$, the corresponding complex triad $A_k^{(r)+}$ undergoes a definite three-dimensional complex rotation which carries it onto $B_k^{(r)+}$ and the correspondence between the connected Lorentz group $S\mathfrak{L}_4$ on $a_\mu^{(\xi)}$ and the complex rotation group SO_3^* on $A_k^{(r)+}$ is an *isomorphism.* Of course the same isomorphism happens for the three-dimensional rotation group on $A_k^{(r)-}$ in the space E_3^-.

This important statement allows us to introduce the hypersphere S_3^+ in the four-dimensional complex Euclidean space E_4^+ as the manifold of the group SO_3^*, that is, as configuration space for our internal states, and to extend the preceding nonrelativistic theory to the relativistic case by Cartan's "complexification" procedure.

Indeed each relative orientation of the T frame with respect to the L frame, that is, each internal configuration of the structure, may be expressed by the two (conjugate) three-dimensional complex rotations which correspond to the Lorentz transform under consideration; that is, finally by a pair of opposite points $(z_\mu, -z_\mu)$ on the complex hypersphere S_3^*.

As regards the invariance of the theory, we shall first assume, as an abstract generalization of the above nonrelativistic treatment, the formalism to be invariant under independent Lorentz transforms acting on the fixed and moving tetrads. This amounts to consider as invariance transformations the right and left translations on the three-dimensional rotation group, that is the bilateral group $\mathrm{Bil}(SO_3^*)$. In the same way as above, this bilateral group is isomorphic to $SO_3^* \times SO_3^*$.

We have thus the infinitesimal transformations:

$$\delta z_\mu = (\alpha^i L_i + \alpha'^i R_i)_{\mu\nu} z_\nu, \tag{26}$$

where α^i, α'^i are independent infinitesimal *complex* parameters, with the same meaning as above for L_i and R_i. Any function $F(z)$ defined in $\mathfrak{C}_4$ depends both of $\mathrm{Re}(z)$ and of $\mathrm{Im}(z)$. But we can also consider as independent variables the two complex conjugate variables z_μ and z_μ^*, which we may write z_μ^+ and z_μ^-, which lie, respectively, in the spaces E_4^+ and E_4^-. We have thus:

$$\delta z_\mu^\pm = (\alpha^{i\pm} L_i + \alpha'^{i\pm} R_i)_{\mu\nu} z_\nu^\pm. \tag{27}$$

If we put:

$$\partial_i^\pm = \partial_\mu^\pm (L_i)_{\mu\nu} z_\nu^\pm, \quad \partial_i'^\pm = \partial_\mu^\pm (R_i)_{\mu\nu} z_\nu^\pm, \tag{28}$$

we get

$$\delta F(z^+,z^-) = (\alpha^{i+}\partial_i^+ + \alpha^{i-}\partial_i^- + \alpha'^{i+}\partial_i'^+ + \alpha'^{i-}\partial_i'^-) F(z^+,z^-). \tag{29}$$

We then introduce the quantum operators:

$$J_i^\pm = (\hbar/i)\partial_i^\pm, \quad J_i'^\pm = (\hbar/i)\partial_i'^\pm, \quad J^{2\pm} = J_i^\pm J_i^\pm = J_i'^\pm J_i'^\pm, \tag{30}$$

with the relations[7]:

[7] See reference 6.

$$[J_i^\pm, J_j^\pm] = (\hbar/i)\epsilon_{ijk} J_k^\pm, \quad [J_i'^\pm, J_j'^\pm] = (\hbar/i)\epsilon_{ijk} J_k'^\pm,$$
$$[J_i^\pm, J_j'^\pm] = [J_i^\pm, J_j^\mp] = [J_i'^\pm, J_j'^\mp] = [J_i^\pm, J_j'^\mp] = 0,$$
$$[J^{2\pm}, J_i^\pm] = [J^{2\pm}, J_i'^\pm] = [J^{2\pm}, J_i^\mp] = [J^{2\pm}, J_i'^\mp] = 0. \tag{31}$$

As a consequence, we can consider six commuting operators, namely $J^{2\pm}$, $J_3^\pm$, $J_3'^\pm$, and seek their common eigenfunctions. As the manifold is not compact, we know we get first a continuous spectrum corresponding to the infinite-dimensional representations of the group. Making the plausible assumption these representations will play no role in our theory which considers only stable quantum states, we shall exclude these functions. This entails the drawback that the remaining discrete eigenfunctions no longer constitute a complete set for all the functions $F(z^+,z^-)$. But it can be shown that they do form a complete set for the functions of the form:

$$F(z^+,z^-) = P(z^+)\cdot P'(z^-), \tag{32}$$

where P and P' are polynomials. In the present theory, we shall restrict ourselves to such functions. Now the discrete eigenfunctions have the form:

$$U(l^+,l^-; m^+,m'^+,m^-,m'^-)(z^+,z^-) = Y(l^+; m^+,m'^+)(z^+)\cdot Y(l^-; m^-,m'^-)(z^-), \tag{33}$$

where $Y(l^\pm; m^\pm, m'^\pm)$ have exactly the same form as the generalized spherical functions $Y(l; m,m')$ considered in the nonrelativistic case. One knows, moreover, that l^+ and l^- take independently integer or half-integer values, $m^\pm$ and $m'^\pm$ lying in the sets

$$-l^\pm, -l^\pm+1, \cdots, l^\pm-1, l^\pm.$$

The construction of a Hilbert space with these functions raises some difficulties, as the configuration space S_3^* is not compact and therefore allows no converging integration. Nevertheless, Souriau and one of us have recently suggested an indirect way of computing an invariant measure for the polynomials in $z_\mu^\pm$.[8] Let us consider the functions of the rotation, defined on $\mathfrak{s}_3^*$, and extend them analytically to the whole space $\mathfrak{C}_4$. The eigenfunctions of J^{2+}, which are, in fact, the spherical functions on S_3^*, then appear as the trace on S_3^* of the harmonic polynomials $P(z_\mu^+)$ in $\mathfrak{C}_4$. Moreover, each combination $F(z_\mu^+)$ of these polynomials $P(z_\mu^+)$ defined on S_3^* may be endowed in an unique way with a *harmonic extension* $\tilde{F}(z_\mu^+)$. This extension satisfies

$$\Delta^+ \tilde{F}(z_\mu^+) = 0, \tag{34}$$

in the whole space $\mathfrak{C}_4$, and

$$\tilde{F}(z_\mu^+) = F(z_\mu^+) \quad \text{on} \quad S_3^*. \tag{35}$$

Now if we consider the value $\tilde{F}(0)$ of $\tilde{F}(z_\mu^+)$ at the center of the sphere, we know it provides a measure of $\tilde{F}(z_\mu^+)$ which is invariant under the rotations in $\mathfrak{C}_4$, so that we can write by definition:

$$\langle F(z_\mu^+)\rangle = \tilde{F}(0). \tag{36}$$

This is immediately generalized to the eigenfunctions common to J^{2+} and J^{2-} which are products of polynomials:

$$F(z^+,z^-) = P(z^+)\cdot P'(z^-),$$

each of them being harmonic, respectively, with respect to the two operators:

$$\Delta^+ = \partial^2/\partial z_\mu^+ \partial z_\mu^+, \quad \Delta^- = \partial^2/\partial z_\mu^- \partial z_\mu^-,$$

so that the product is harmonic with respect to $\Delta = \Delta^+ + \Delta^-$. We thus can define the harmonic extension

$$\tilde{F}(z^+,z^-) = \tilde{P}(z^+)\cdot \tilde{P}(z^-), \tag{37}$$

and consequently the invariant measure

$$\langle F(z^+,z^-)\rangle = \tilde{F}(0,0). \tag{38}$$

It is now easy to see that:

$$\langle P(z^+)\cdot P'(z^-)\rangle = \langle P(z^+)\rangle \cdot \langle P'(z^-)\rangle. \tag{39}$$

[8] F. Halbwachs and J. M. Souriau (to be published).

This measure coincides in the real case with the ordinary measure obtained by integration of $F(y)$ over the compact sphere S_3. This leads to a definition of scalar product in our functional (pseudo-Hilbert) space, namely, the invariant measure:

$$(F_1,F_2)=\langle F_1{}^*\cdot F_2\rangle=\langle F_1{}^*\rangle\cdot\langle F_2\rangle. \tag{40}$$

The use of the complex conjugation, which, of course, acts both on the functional and on the variables, is justified by the fact that the complex conjugation is the only operation which commutes with the complex rotations. With respect to our eigenfunctions, we get:

$$\langle U^*(l^+,l^-;m^+,m'^+,m^-,m'^-)| \\ =\langle U(l^-,l^+;-m^-,-m'^-,-m^+,-m'^+)|, \tag{41}$$

so that the bra-ket transition transforms the polynomials in z^+ into polynomials in z^-; so that certain eigenfunctions have their norm equal to zero.

With respect to this indefinite metric, our basic operators are "pseudo-Hermitian"; they do not obey the usual Hermiticity conditions

$$(F^*,AG)=(\{AG\}^*,F). \tag{42}$$

Nevertheless, they have real eigenvalues.

On an other hand, one can consider the set of common eigenfunctions of the Hermitian conjugates of our basic operators, which form another set of functions $V(l^+,l^-;m^+,m'^+,m^-,m'^-)(z^+,z^-)$. Now we can check by forming the scalar products (U,V) that each function $U(l^+,l^-;m^+,m'^+,m^-,m'^-)$ associated with given values of l^+, l^-, m^+, m^-, m'^+, m'^-, is orthogonal to all functions V associated with other values of these quantum numbers, so that the splitting of any function $F(z^+,z^-)=P(z^+)$. $P'(z^-)$ on the set of functions U is unique. All the questions related to this pseudo-Hilbert space are treated in detail in a paper to be published by Souriau and one of us.[8]

The functional space $\mathcal{H}$ of the polynomials of the form $F(z^+,z^-)=P(z^+)$. $P'(z^-)$ is then spanned by the set of "pseudo-orthogonal" functions $U(l^+,l^-;m^+,m'^+,m^-,m'^-)$. Moreover, it is obvious, if we consider all the functions U with the same given values of l^+, l^-, that they span inside $\mathcal{H}$ a *subspace* $H(l^+,l^-)$ which is invariant under Bil$SO_3{}^*$ and transforms under the irreducible representation $D(l^+,l^-)$ of $SO_3{}^*\times SO_3{}^*$.

Finally, if we consider more rigorously the set of functions U with the same given values of l^+, l^-, m'^+, m'^-, they span inside $H(l^+,l^-)$ a subspace, called the *level* $E(l^+,l^-,m'^+,m'^-)$, which is invariant under the *left* rotations $SO_3{}^*$ and transforms under the irreducible representation $D(l^+,l^-)$ of the group $SO_3{}^*$.

Naturally, the assumption that transformations from the $a_\mu{}^{(\xi)}$ to the $b_\mu{}^{(\xi)}$ frames are related to real possible motions restricts us to *proper* Lorentz transforms. On the contrary, as regards the transformations on the tetrads leaving the formalism invariant, we are allowed to make inversions acting on both tetrads together. Such an inversion transforms the proper Lorentz transform Λ_z into another proper Lorentz transform.

Let us first consider the *space inversion*, which applies to both tetrads the matrix:

$$g=\begin{bmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{bmatrix}. \tag{43}$$

We have:

$$\Lambda_{(Pz)}=g\Lambda_2 g^{-1}. \tag{44}$$

The external automorphism $z\rightarrow Pz$ is not included in the group Bil$SO_3{}^*$. It is easy to show that:

$$Pz^+ + z^-,\quad Pz^-=z^+. \tag{45}$$

The inversion P induces in the functional space $\mathcal{H}$ of the polynomials the transformation defined by:

$$PU(l^+,l^-;m^+,m'^+,m^-,m'^-) \\ =(-1)^{l^++l^-+m'^++m'^-}\cdot U(l^-,l^+;m^-,m'^-,m^+,m'^+). \tag{46}$$

The factor ± 1 is chosen for reasons to be explained later, (the P conjugated functions being defined to an arbitrary constant coefficient).

Further, the *time reversal* employs the matrix $-g$, and thus gives the same transformation $Tz^+\rightarrow z^-$, $Tz^-\rightarrow z^+$. But it can be shown that the correct quantization leads, as regards the derivatives, to

$$T(\partial/\partial z^\pm)=-\partial/\partial z^\mp, \tag{47}$$

so that the T conjugation transforms $J_k{}^\pm$ and $J_k{}'^\pm$ into $-J_k{}^\mp$ and $-J_k{}'^\mp$ and induces in the space $\mathcal{H}$ the transformation

$$TU(l^+,l^-;m^+,m'^+,m^-,m'^-) \\ =(-1)^{l^++l^--m^+-m^-} \\ \times U(l^-,l^+;-m^-,-m'^-,-m^+,-m'^+). \tag{48}$$

Finally the *total inversion*, that is the $P\cdot T$ operation, which simply changes the sign of $z^\pm$, induces in the space $\mathcal{H}$ the transformation (C conjugation):

$$CU(l^+,l^-;m^+,m'^+,m^-,m'^-) \\ =(-1)^{l^++l^-+|m'^++m'^--m^+-m^-|} \\ \times U(l^+,l^-;-m^+,-m'^+,-m^-,-m'^-). \tag{49}$$

Thus the P conjugation, and also the T conjugation, transfer us from the subspace $H(l^+,l^-)$ to the subspace $H(l^-,l^+)$, while the C conjugation leaves the subspace $H(l^+,l^-)$ invariant but transfers us from the *level* $E(l^+,l^-;m'^+,m'^-)$ into the level $E(l^+,l^-;-m'^+,-m'^-)$.

Let us now come back to our invariance statement. We intend to generalize in a suitable relativistic way the invariance assumed for the nonrelativistic model. The general nonrelativistic rotation invariance gives rise in relativistic physics to the Lorentz invariance, so that we allow all Lorentz transforms acting on both tetrads together. But the spherical symmetry of the nonrelativistic structure cannot be extended in four dimensions, since the vector $b_\mu^{(4)}$ plays a particular role as a four-velocity and cannot be chosen arbitrarily. Consequently, we have to deal with arbitrary Lorentz transforms on both tetrads considered as a block, and in addition with arbitrary three-dimensional rotations on the spacelike vectors of the moving tetrad. Or, equivalently, we have Lorentz transforms on the left and spatial rotations on the right: If Λ_z designates a Lorentz transform related to the point z on S_3^*, we have the transformation

$$\Lambda_{z'}=\Lambda_{z1}\cdot\Lambda_z\begin{pmatrix}1 & 0\\ 0 & \Omega_y\end{pmatrix}^{-1}, \tag{50}$$

where Ω_y is any 3×3 spatial real rotation matrix.

Thus we deal in fact not with the bilateral group $\mathrm{Bil}S\mathfrak{L}_4=\mathrm{Bil}SO_3^*$ studied above, but with a subgroup of it, obtained by restricting the right translations on the group to spatial rotations. This is finally our actual invariance group G which is isomorphic to $S\mathfrak{L}_4\times SO_3$, or equivalently to $SO_3^*\times SO_3$.

The condition for a Lorentz transform Λ_{z2} to be real, that is to restrict to the form $\begin{pmatrix}1 & 0\\ 0 & \Omega\end{pmatrix}$, with Ω real, is equivalent to taking

$$z_2^+=z_2^-=\tfrac{1}{2}(z_2^++z_2^-). \tag{51}$$

Now our infinitesimal transformations become

$$\delta z_\mu^\pm=\left(\alpha^{i\pm}L_i+\frac{\alpha'^{i+}+\alpha'^{i-}}{2}R_i\right)_{\mu\nu}z_\nu^\pm, \tag{52}$$

$$\delta F(z^+,z^-)=[\alpha^{i+}\partial_i^++\alpha^{i-}\partial_i^-+\alpha'^i(\partial_i'^++\partial_i'^-)]\times F(z^+,z^-), \tag{53}$$

where $\alpha'^i=\frac{1}{2}(\alpha'^{i+}+\alpha'^{i-})$ is real.

We are thus led to introduce, instead of $J_k'^+$ and $J_k'^-$ the operators

$$S_k'=J_k'^++J_k'^-, \tag{54}$$

and consequently the operator

$$S'^2=S_k'S_k'. \tag{55}$$

We have:

$$[S_i',S_j']=(\hbar/i)\epsilon_{ijk}S_k',\quad [S_i',J_k^\pm]=0, \tag{56}$$

and S'^2 commutes with $J^{2\pm}$, $J_k^\pm$, S_k'. We can consider once more six commuting operators: $J^{2\pm}$, $J_3^\pm$, S'^2, and S_3' and seek their common eigenfunctions which will have the form $Z(l^+,l^-;s';m^+,m^-,m')$, with

$$\begin{aligned} J^{2\pm}Z&=l^\pm(l^\pm+1)\hbar^2Z, & S'^2Z&=s'(s'+1)\hbar^2Z,\\ J_3^\pm Z&=m^\pm\hbar Z, & S_3'Z&=m'\hbar Z.\end{aligned} \tag{57}$$

s' lies in the series l^++l^-, l^++l^--1, $\cdots$ $|l^+-l^-|$ and m' in the series $-s'$, $-s'+1\cdots s'-1$, s'.

These results are established by a well-known procedure on the addition of the spin operators, which yields the Z functions as linear combinations of the above U functions, with the aid of Clebsch-Gordan coefficients, namely,[9]

[9] C. van Winter, thesis, Groningen, 1957 (unpublished).

$$Z(l^+,l^-,s';m^+,m^-,m')=\sum_{m'^+,m'^-}(l^+,l^-,m'^+,m'^-|s',m')\times U(l^+,l^-;m^+,m^-,m'^+,m'^-). \tag{58}$$

The Z functions with the same given values for l^+ and l^- lie in the same invariant subspace $H(l^+,l^-)$ as the corresponding U functions. But the Z functions with the same given values for l^+, l^-, s', m' split this subspace into new kinds of levels $\mathcal{E}(l^+,l^-,s';m')$ invariant under the *left* rotation group SO_3^*, which transform under the irreducible representation $D(l^+,l^-)$ of this group.

On these new functions, we have the inversions:

$$PZ(l^+,l^-,s';m^+,m^-,m')=(-1)^{l^++l^--s'}Z(l^-,l^+,s';m^-,m^+,m'), \tag{59}$$

$$TZ(l^+,l^-,s';m^+,m^-,m')=(-1)^{m^++m^--m'}Z(l^-,l^+,s';-m^-,-m^+,-m'), \tag{60}$$

$$CZ(l^+,l^-,s';m^+,m^-,m')=(-1)^{l^++l^--s'+|m^++m^--m'|}\times Z(l^+,l^-,s';-m^+,-m^-,-m'), \tag{61}$$

so that we have:

$$PH(l^+,l^-)=TH(l^+,l^-)=H(l^-,l^+), \tag{62}$$

and

$$C\mathcal{E}(l^+,l^-,s';m')=\mathcal{E}(l^+,l^-,s';-m'). \tag{63}$$

As stated before, let us now classify our internal eigenstates in connection with the experimental elementary particle classification.

Following three of us,[10] we accept the usual assumption that the two charges conserved in all interactions, namely, the baryon number N and the electric charge q, are related to two operators B and Q (acting on each eigenfunction $Z(l^+,l^-,s';m^+,m^-,m')$ associated with two gauge transforms

$$B_{\mathrm{op}}Z(l^+,l^-,s';m^+,m^-,m')=e^{iN\alpha}Z(l^+,l^-,s';m^+,m^-,m'), \tag{64}$$

$$Q_{\mathrm{op}}Z(l^+,l^-,s';m^+,m^-,m')=e^{iq\beta}Z(l^+,l^-,s';m^+,m^-,m'), \tag{65}$$

$N\alpha$ and $q\beta$ being real quantities.

This assumption facilitates the identification of the quantum numbers. We have three operators which perform such gauge transformations, namely, $J_3'^++J_3'^-$, which is equivalent, as we have seen, to a real rotation of the $B_k^{(r)\pm}$ frame, and $J_3^++J_3^-$, which is similarly equivalent to a real rotation of the $A_k^{(r)\pm}$ frame, and naturally their sum or difference which both correspond to multiplication by $\exp(i\lambda)$, with λ real.

[10] D. Bohm, P. Hillion, and J. P. Vigier, Progr. Theoret. Phys. (Kyoto) 24, 701 (1960).

TABLE I. Classification of particles (levels).

Representation	$-m'=\frac{1}{2}N$	$m^+=I_3$	$m^-=\frac{1}{2}S$	$Q=m^++m^-+m'$	Particle	Levels
$D(\frac{1}{2},0)$, $m'=\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	-1	e^-	$Z(\frac{1}{2}, 0, \frac{1}{2}; -\frac{1}{2}, 0, \frac{1}{2})$
	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	ν_e	$Z(\frac{1}{2},0,\frac{1}{2}; \frac{1}{2},0,\frac{1}{2})$
$D(\frac{1}{2},0)$, $m'=-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\bar{e}^-=e^+$	$Z(\frac{1}{2}, 0, \frac{1}{2}; \frac{1}{2}, 0, -\frac{1}{2})$
	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\bar{\nu}_e$	$Z(\frac{1}{2}, 0, \frac{1}{2}; -\frac{1}{2}, 0, -\frac{1}{2})$
$D(0,\frac{1}{2})$, $m'=\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1	μ^-	$Z(0, \frac{1}{2}, \frac{1}{2}; 0, -\frac{1}{2}, \frac{1}{2})$
	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	ν_μ	$Z(0,\frac{1}{2},\frac{1}{2}; 0,\frac{1}{2},\frac{1}{2})$
$D(0,\frac{1}{2})$, $m'=-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\bar{\mu}^-=\mu^+$	$Z(0, \frac{1}{2}, \frac{1}{2}; 0, \frac{1}{2}, -\frac{1}{2})$
	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\bar{\nu}_\mu$	$Z(0, \frac{1}{2}, \frac{1}{2}; 0, -\frac{1}{2}, -\frac{1}{2})$
$D(1,0)$, $m'=0$	0	1	0	1	π^+	$Z(1,0,1; 1,0,0)$
	0	0	0	0	π^0	$Z(1,0,1; 0,0,0)$
	0	-1	0	-1	$\pi^-=\pi^+$	$Z(1, 0, 1; -1, 0, 0)$
$D(\frac{1}{2},\frac{1}{2})$, $s'=0$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	K^-	$Z(\frac{1}{2}, \frac{1}{2}, 0; -\frac{1}{2}, -\frac{1}{2}, 0)$
	0	$\frac{1}{2}$	$\frac{1}{2}$	1	K^+	$Z(\frac{1}{2},\frac{1}{2},0; \frac{1}{2},\frac{1}{2},0)$
	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\bar{K}^0$	$Z(\frac{1}{2}, \frac{1}{2}, 0; \frac{1}{2}, -\frac{1}{2}, 0)$
	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	K^0	$Z(\frac{1}{2}, \frac{1}{2}, 0; -\frac{1}{2}, \frac{1}{2}, 0)$
$D(\frac{1}{2},1)$, $s'=\frac{1}{2}$, $m'=-\frac{1}{2}$ (particles)	$\frac{1}{2}$	$\frac{1}{2}$	1	2	X^{++}	$Z(\frac{1}{2}, 1, \frac{1}{2}; \frac{1}{2}, 1, -\frac{1}{2})$
	$\frac{1}{2}$	$-\frac{1}{2}$	1	1	X^+	$Z(\frac{1}{2}, 1, \frac{1}{2}; -\frac{1}{2}, 1, -\frac{1}{2})$
	$\frac{1}{2}$	$\frac{1}{2}$	0	1	p	$Z(\frac{1}{2}, 1, \frac{1}{2}; \frac{1}{2}, 0, -\frac{1}{2})$
	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	n	$Z(\frac{1}{2}, 1, \frac{1}{2}; -\frac{1}{2}, 0, -\frac{1}{2})$
	$\frac{1}{2}$	$\frac{1}{2}$	-1	0	Ξ^0	$Z(\frac{1}{2}, 1, \frac{1}{2}; \frac{1}{2}, -1, -\frac{1}{2})$
	$\frac{1}{2}$	$-\frac{1}{2}$	-1	-1	Ξ^+	$Z(\frac{1}{2}, 1, \frac{1}{2}; -\frac{1}{2}, -1, -\frac{1}{2})$
$D(1,\frac{1}{2})$, $s'=\frac{1}{2}$, $m'=-\frac{1}{2}$ (particles)	$\frac{1}{2}$	1	$\frac{1}{2}$	2	Y^{++}	$Z(1, \frac{1}{2}, \frac{1}{2}; 1, \frac{1}{2}, -\frac{1}{2})$
	$\frac{1}{2}$	0	$\frac{1}{2}$	1	Y^+	$Z(1, \frac{1}{2}, \frac{1}{2}; 0, \frac{1}{2}, -\frac{1}{2})$
	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	Y^0	$Z(1, \frac{1}{2}, \frac{1}{2}; -1, \frac{1}{2}, -\frac{1}{2})$
	$\frac{1}{2}$	1	$-\frac{1}{2}$	1	Σ^+	$Z(1, \frac{1}{2}, \frac{1}{2}; 1, -\frac{1}{2}, -\frac{1}{2})$
	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	Σ^0	$Z(1, \frac{1}{2}, \frac{1}{2}; 0, -\frac{1}{2}, -\frac{1}{2})$
	$\frac{1}{2}$	-1	$-\frac{1}{2}$	-1	Σ^-	$Z(1, \frac{1}{2}, \frac{1}{2}; -1, -\frac{1}{2}, -\frac{1}{2})$

Now we can tentatively identify the operator $J_3'^+ + J_3'^- = S_3'$ with $-\frac{1}{2}B_{\text{op}}$, and the operator $J_3^+ + J_3^- - S_3'$ with Q_{op}, as two of the three operators providing gauge transformations $\exp(i\lambda)$, where λ is a real argument.

With these conventions we see that the baryon gauge corresponds to $-S_3'$, with integer (or half-integer) eigenvalues $-m'=N/2$, the integer N being the usual baryon (fermion) number. The Pauli electric gauge is then obtained by the action of:

$$Q_{\text{op}} = J_3^+ + J_3^- - S_3', \qquad (66)$$

whose eigenvalues (always integer) are

$$-m' + m^+ + m^- = q. \qquad (67)$$

As we shall see later, these operators J_3^+, J_3^-, S_3', and Q commute with the internal Hamiltonian and are constants of the motion.

This leads us very naturally to identify J_3^+ with $I_{3\text{op}}$ and J_3^- with $\frac{1}{2}S$ since we know that their eigenvalues must be constants of the motion. This identification yields the expression

$$Q_{\text{op}} = I_3 + \frac{1}{2}S + \frac{1}{2}B, \qquad (68)$$

which justifies the Nishijima–Gell-Mann formula, and we see that the corresponding "level" or "particle" classification recovers the Nishijima–Gell-Mann empirical scheme. (See Table I) in which each line gives the quantum numbers associated with a given "level" and its corresponding element in the Nishijima–Gell-Mann empirical table.[11a]

At this stage, we shall only stress the following features of Table I:

(a) It ascribes isobaric spins and strangeness to leptons, thus extending the Nishijima–Gell-Mann scheme.

(b) It contains two kinds of neutrinos associated, respectively, with e^- and μ^-, the existence of which is presently assumed by Pontecorvo[11] and various authors.

(c) It contains an (as yet) unobserved doublet in $D(\frac{1}{2},1)$ and triplet in $D(1,\frac{1}{2})$. As we shall see later, the fact that these multiplets are the only ones which

[11a] *Note added in proof.* If one accepts instead of this scheme the proposal of Yukawa (see Appendix of Paper II) one should write p, n, v, and Λ instead of e^+, $\bar{r}_e$, μ^+, and $\bar{r}_\mu$ (in $D(\frac{1}{2},0)$ and $D(0,\frac{1}{2})$) in Table I; N of $D(\frac{1}{2},1)$ becoming a resonance. The four basic leptons e^-, r_e, μ^-, and r_μ are then introduced in the representations $D'(\frac{1}{2},0)$ and $D'(0,\frac{1}{2})$ of the group $G'=SO_3\times SO_3'^+\times SO_3'^-$ obtained by interchanging the role of L and T.

[11] B. Pontecorvo, J. Exptl. Theoret Phys. (U.S.S.R.) 37, 1751 (1959).

contain a charge 2 particle may account for their instability.

(d) The Λ^0 does not appear in this table, but, as we shall discuss in a following paper, it can be identified with a isobaric spin singlet in the $D(0,\frac{1}{2})$ subspace, provided that some external property differentiates it from leptons.

(e) The table is not exclusive. Even if we limit ourselves to representations with $l^{\pm}\leq 1$, we can find "levels" belonging to $D(1,0)$, $D(0,1)$ or $D(\frac{1}{2},\frac{1}{2})$ with $m'=\pm 1$. This yields the possibility of highly excited states, some of which could correspond to recently observed resonances.

CONCLUSION

We wish to conclude this first paper (which is based essentially on pure group-theoretical considerations) with some remarks on the geometrical meaning of the variables and operators.

As was shown by Synge,[12] van Winter,[9] and the first approaches of the authors,[13] the Lorentz transform from fixed to moving frames can be performed by a series of elementary transformations, namely:

(1) spatial rotations which are represented by real Euler angles φ_1, θ_1, ψ_1;

(2) pure Lorentz transformations which can be represented by hyperbolic rotations from one spacelike vector to the timelike vector, and which use imaginary arguments $i\varphi_2$, $i\theta_2$, $i\psi_2$.

It is possible to perform these six transformations in a suitable order, such as to realize any particular Lorentz transformation. These six generalized Euler angles are then a special kind of parameters labeling the Lorentz transform. Now it can be shown that, if we consider the complex conjugate combination

$$\varphi^{\pm}=\varphi_1\pm i\varphi_2,\quad \theta^{\pm}=\theta_1\pm i\theta_2,\quad \psi^{\pm}=\psi_1\pm i\psi_2, \qquad (68)$$

the three parameters φ^+, θ^+, ψ^+ represent the complex three-dimensional Euler angle of the rotation of $B_k^{(r)+}$ with respect to $A_k^{(r)+}$ in E_3^+, so that they can be considered as coordinates of the figurative point P on the complex hypersphere S_3^*. The same holds for the three complex angles φ^-, θ^-, ψ^- in the space E_3^-.

Now in the above formalism each coordinate y_μ of point P splits into a real part related to a real rotation and an imaginary part related to a pure Lorentz transform. In particular each of the basic operators $J_k^{\pm}$ is related to an infinitesimal space-rotation and Lorentz transform in a perpendicular direction.

More precisely, it can be shown the chosen third operators are related in a very simple way to the complex Euler angles, namely,

$$J_3^{\pm}=(\hbar/i)\partial/\partial\varphi^{\pm},\quad J_3'^{\pm}=(\hbar/i)\partial/\partial\psi^{\pm}, \qquad (70)$$

so that, for instance, $J_3^{\pm}$ represents together two infinitesimal transformations, namely, a rotation in the plane $a_\mu^{(1)}$, $a_\mu^{(2)}$ and a pure Lorentz transform along the vector $a_\mu^{(3)}$. In the same way $J_3'^{\pm}$ implies a rotation in $b_\mu^{(1)}$, $b_\mu^{(2)}$ and a pure Lorentz transform along $b_\mu^{(3)}$.

The quantization introduced simultaneously on the real and imaginary parts of the complex Euler angles is therefore quite natural. It can evidently be interpreted as expressing the very plausible statement that elementary particles correspond to internal motions in which the T and L frames come back periodically to the same relative orientation. In another form, as indicated before by one of us (L. d. B.) this is just a concrete representation of the "clock" attached by wave mechanics to every material element.

APPENDIX I

As stated in our introduction, we shall briefly discuss in this Appendix the question of the interpretation of our model from three different points of views.

(A) The first point of view considering elementary particles as extended structures starts from a classical extended model and quantizes it along the usual lines. When we do this, however, a well-known stumbling block appears immediately: The transition from a classical pointlike particle to an extended structure necessarily introduces an infinite number of new "internal" degrees of freedom corresponding to the extension of the particle in space. In order to treat mathematically such an internal fieldlike structure one must therefore, as was proposed by some of us in a detailed study of the classical level,[14] abstract out of this infinity a finite number of average collective kinematical variables (which correspond to essential characteristics of the internal motion) and determine the internal energy. Such a procedure, however, raises many difficult problems (especially in the relativistic domain); and it will be discussed in another paper.

Now one way around this stumbling block exists, which has already been introduced by the authors[15]; namely, to describe *a priori* this average internal motion by a finite number of new kinematical variables $q(\tau)$, which are added to the usual kinematical position variables $x_\mu(\tau)$ (τ being the proper time along the world line followed by the particle's center x_μ) associated with the classical point particle. In this procedure, which we shall now develop in some detail, one must keep in mind two essential points:

(a) It is necessary to be very careful to use only

[12] J. L. Synge, *Relativity, the Special Theory* (North-Holland Publishing Company, Amsterdam, 1956).

[13] F. Halbwachs, P. Hillion, and J. P. Vigier, Ann. Inst. Henri Poincaré **16**, 115 (1959).

[14] D. Bohm, P. Hillion, T. Takabayasi, and J. P. Vigier, Progr. Theoret. Phys. (Kyoto) **23**, 496 (1960).

[15] F. Halbwachs, J. M. Souriau, and J. P. Vigier, J. Phys. Radium **22**, 393 (1961).

independent kinematical variables since, after quantization, the existence of a total wave function $\phi(x_\mu,q,\tau)$ describing internal and external motions implies the existence of a representation in which x_μ and q are simultaneously diagonal. This point has been stressed in particular by Pryce[16] and one of us.[17]

(b) All these new variables q are assumed to be functions of the proper time τ of a single point $x_\mu(\tau)$, so that we avoid all causality troubles and many-time problems occurring in preceding theories.

As indicated in Sec. I, in the nonrelativistic domain, where τ reduces to the ordinary time t, one can add to the classical external point Lagrangian

$$L_{(e)}=\tfrac{1}{2}m\dot{x}_k\dot{x}_k \tag{71}$$

(where $\dot{A}$ denotes dA/dt), an internal Lagrangian:

$$L_{(i)}=\tfrac{1}{2}I\Omega_k\Omega_k, \tag{72}$$

where I is a moment of inertia and

$$\Omega_k=\tfrac{1}{2}\epsilon_{ijk}\dot{b}_i^{(r)}b_j^{(r)} \tag{73}$$

represents the instantaneous rotation of the T frame $b_k^{(r)}$ with respect to the L frame $a_k^{(r)}$.[18] The motion of the extended particle thus represented by two frames (L and T) centered on a moving point can be quantized by the usual method.

If we start from the classical point of view, we know that the evolution of our internal state can be represented as the motion of a point P along a line $\mathfrak{L}$ on the Riemannian manifold S_3. Now, as a calculation shows (the Riemannian metric being taken into account), the internal rotation kinetic energy

$$T=\tfrac{1}{2}I\Omega_k\Omega_k=\tfrac{1}{2}I\dot{b}_k^{(r)}\dot{b}_k^{(r)}, \tag{74}$$

if expressed in terms of the coordinates y_μ of P on S_3, is simply:

$$T=2I\dot{y}_\mu\dot{y}_\mu=2I(ds/dt)^2, \tag{75}$$

where ds is the element of the line $\mathfrak{L}$ on the hypersphere. This is highly analogous to the usual expression for the translation kinetic energy, $T=\frac{1}{2}m(ds/dt)^2$, where ds is the element of trajectory in Euclidean space.

From this analogy the mechanical quantization is easily deduced. We have the internal canonical momenta

$$p_\mu=\partial T/\partial\dot{y}_\mu=4I\dot{y}_\mu \tag{76}$$

and the classical Hamiltonian

$$H_{(i)}=(1/8I)p_\mu p_\mu. \tag{77}$$

The quantization replaces p_μ by $(\hbar/i)(\partial/\partial y_\mu)$. Finally the quantum Hamiltonian is

$$H_{(i)\mathrm{op}}=-(\hbar^2/8I)(\partial^2/\partial y_\mu\partial y_\mu), \tag{78}$$

the relation $y_\mu y_\mu=1$ being taken into account, and it is immediately shown it has just the expression

$$H_{(i)\mathrm{op}}=(1/2I)J^2, \tag{79}$$

so that we immediately recover all the preceding group-theoretical treatment.

Replacing moreover the "external" canonical momenta

$$G_k=\partial L/\partial\dot{x}_k \quad \text{by} \quad (\hbar/i)(\partial/\partial x_k), \tag{80}$$

we see that the generalized Schrödinger equation, associated with the Lagrangian

$$L=L_{(e)}+L_{(i)}, \tag{81}$$

becomes

$$-i\hbar\partial\Phi/\partial t=H_{\mathrm{op}}\Phi=-(\hbar^2/2m)\Delta\Phi-(1/2I)J^2\Phi, \tag{82}$$

where Φ is a function of x_k, y_μ, and t. The internal stationary solutions of (82) are evident. Writing

$$\Phi(x_k,y_\mu,t)=\exp(-iE_{(i)}t/\hbar)\varphi_e(x_k,t)F(y_\mu), \tag{83}$$

$\varphi_e(x_k,t)$ and $F(y_\mu)$ being external and internal state functions associated to the internal energy $E_{(i)}$, we see that relation (82) splits into an external equation

$$(-i\hbar\partial/\partial t+(\hbar/2m)\Delta)\varphi_e(x_k,t)=0, \tag{84a}$$

and an internal equation

$$(J_kJ_k-2E_{(i)}I)F(y_\mu)=0, \tag{84b}$$

invariant under our internal group $G=\mathrm{Bil}(SO_3)$. Relation (84b) is evidently the internal counterpart of the external Schrödinger equation (84a) (which is invariant under the Galilean group) and defines stationary internal waves which determine internal quantum states of the structure of the particles. Moreover, any solution F can be developed in terms of the functions $Y(l;m,m')(y_\mu)$. We shall have $F(y_\mu)=Y(l;m,m')$ only if

$$2E_{(i)}I=\hbar^2l(l+1). \tag{85}$$

The corresponding relativistic model proposed by one of us (JPV) also rests on the idea that the extended particle can be represented by two kinematical frames $L(a_\mu^{(\xi)})$ and $T(b_\mu^{(\xi)})$ centered on a moving point $x_\mu(\tau)$.

Let us first recall certain well-known results of the relativistic point-particle theory. If we define the particle's position in Minkowski space by four kinematical parameters $x_\mu(\tau)$, τ being as before the proper time along the world line followed by x_μ, we can define its path by the "line" Lagrangian

$$L_{(e)}=\tfrac{1}{2}m(\dot{x}_\mu\dot{x}_\mu+c^2) \tag{86}$$

(m being a Lagrange multiplier corresponding to the constraint $\dot{x}_\mu\dot{x}_\mu=-c^2$), and the external relativistic action function $W_{(e)}$ with

$$W_{(e)}=\int_{\tau_1}^{\tau_2}L_{(e)}d\tau, \tag{87}$$

[16] M. H. Pryce, Proc. Roy. Soc. (London) A195, 62 (1948).
[17] T. Takabayasi, Progr. Theoret. Phys. (Kyoto) 25, 901 (1961).
[18] D. Bohm, P. Hillion, and J. P. Vigier, Progr. Theoret. Phys. (Kyoto) 24, 701 (1960).

the real motion corresponding to the path

$$\delta W_{(e)}=0. \qquad (88)$$

The corresponding classical and quantum motions are evident. Classically, we introduce the canonical momenta by

$$G_\mu=\partial L_{(e)}/\partial \dot{x}_\mu=m\dot{x}_\mu. \qquad (89)$$

The scalar relativistic Hamiltonian then becomes

$$H_{(e)}=G_\mu\dot{x}_\mu-L_{(e)}=(1/2m)G_\mu G_\mu-\tfrac{1}{2}mc^2. \qquad (90)$$

The corresponding equations of motion then yield immediately

$$\dot{G}_\mu=0 \quad \text{and} \quad G_\mu G_\mu=-c^2=\text{const}; \qquad (91)$$

so that m corresponds to constant rest mass. As a consequence, we see that $H_{(e)}=-mc^2$ is also (as it should) a constant of the motion.

The quantization can be performed along the usual lines. We introduce a general wave function $\Phi(x_\mu,\tau)$, replace $H_{(e)}$ by $-i\hbar\partial/\partial\tau$, and G_μ by $-i\hbar\partial_\mu$, and thus obtain the generalized Schrödinger equation:

$$-i\hbar(\partial\phi/\partial\tau)=[-(\hbar^2/2m)\Box-\tfrac{1}{2}mc^2]\phi. \qquad (92\text{a})$$

Introducing then stationary solutions of the form $\exp(-imc^2\tau/\hbar)\varphi_e(x_\mu)$ into (8), we see that $\varphi_e(x_\mu)$ must satisfy the usual Klein-Gordon equation:

$$(\Box-\mu^2)\varphi_e(x_\mu)=0 \quad \text{with} \quad \mu^2=m^2c^2/\hbar^2. \qquad (92\text{b})$$

The factor $\exp(-imc^2\tau/\hbar)$ is just the "beat" of the "clock" associated by one of us (L. de B.) with every particle since the very beginning of wave mechanics. We know, moreover, that the field $\varphi_e(x_\mu)$ must evidently be invariant under transformations of the Lorentz group $\mathfrak{L}_4$, and, therefore, associated with its finite dimensional representations. We shall discuss this point later in paper II.

Now, as indicated in Sec. I, the transition from pointlike to extended particles can be performed by the introduction in the Lagrangian of new internal variables $q_\alpha^{(\xi)}(\tau)$. As a consequence one discovers, in general, that

$$G_\mu=\partial L/\partial\dot{x}_\mu \qquad (93)$$

is no longer parallel to $\dot{x}_\mu$, so that the rest mass

$$G_\mu\dot{x}_\mu=-mc^2 \qquad (94)$$

is no longer equal to the inertial mass

$$G_\mu G_\mu=-M^2c^2. \qquad (95)$$

Moreover, the point X_μ defined by

$$X_\mu-x_\mu=(1/M^2c^2)S_{\mu\nu}G_\nu \qquad (96)$$

describes, in general, a straight world line parallel to G_μ, $S_{\mu\nu}$ indicating the internal angular momentum:

$$S_{\mu\nu}=[(\partial L/\partial\dot{q}_\mu^{(\xi)})q_\nu^{(\xi)}-(\partial L/\partial\dot{q}_\nu^{(\xi)})q_\mu^{(\xi)}]. \qquad (97)$$

One can show, finally that x_μ spirals around X_μ in a motion constituting the classical counterpart of Schrödinger's Zitterbewegung; X_μ and x_μ playing, respectively the part of Møller's center of mass and of the center of matter density defined by two of us.[19] Evidently[20] the Weyssenhoff equations,[21]

$$\dot{G}_\mu=0 \quad \text{and} \quad \dot{S}_{\mu\nu}=G_\mu\dot{x}_\nu-G_\nu\dot{x}_\mu, \qquad (98)$$

correspond to the invariance of L under the displacement and Lorentz groups.

As proposed by one of us,[22] a first group of internal variables appears immediately. If we assume with two of us (D. B. and J. P. V.) the existence of an *internal conserved current density:*

$$j_\mu=\rho u_\mu \quad (\text{with } \partial_\mu j_\mu=0), \qquad (99)$$

we can introduce, if the particle is small enough, a frame $b_\mu^{(\xi)}$ representing its instantaneous rotation around x_μ. This Takabayasi frame, or T frame, is defined by the well-known relations:

$$\dot{b}_\mu^{(\xi)}b_\nu^{(\xi)}=\omega_{\mu\nu}=\partial_\mu u_\nu-\partial_\nu u_\mu. \qquad (100)$$

Another frame appears immediately if we consider, besides the internal current density, the *internal conserved energy-momentum density:*

$$t_{\mu\nu} \quad (\text{with } \partial_\nu t_{\mu\nu}=0), \qquad (101)$$

for we can take its values at the point x_μ. $t_{\mu\nu}(x_\alpha)$ defines a frame $a_\mu^{(\xi)}$ studied in particular by Lichnerowicz[23] (called L frame for short), defined by the relations:

$$t_{\mu\nu}a_\mu^{(\xi)}=s^{(\xi)}a_\mu^{(\xi)}, \qquad (102)$$

where $s^{(\xi)}$ denote the usual dilatation coefficients. These two frames $a_\mu^{(\xi)}$ and $b_\mu^{(\xi)}$ we tentatively introduce as new internal kinematical variables.

Now the basic physical assumption made here is to assume that the internal Lagrangian depends only on the relative orientation of the T and L frames through the expression $\frac{1}{4}I\omega_{\alpha\beta}\omega_{\alpha\beta}$, where $\omega_{\alpha\beta}$ expresses the relative instantaneous rotation of T with respect to L: a rotation which can be expressed in terms of the complex variables z_μ^+, z_μ^- on the sphere S_3^*. This means that in the L frame we have (writing $\dot{b}_\mu^{(\xi)}$ as functions of $z_\mu^\pm$ so that the orthonormality is automatically satisfied):

$$L=L_{(e)}+L_{(i)}=\tfrac{1}{2}m(\dot{x}_\mu\dot{x}_\mu+c^2) + \tfrac{1}{4}I\omega_{\alpha\beta}\omega_{\alpha\beta}+\alpha^{(r)}b_\mu^{(r)}\dot{x}_\mu, \qquad (103)$$

where m and $\alpha^{(r)}$ are Lagrange multipliers (variable in principle) which imply the relations

$$\dot{x}_\mu\dot{x}_\mu=-c^2 \quad \text{and} \quad b_\mu^{(r)}\dot{x}_\mu=0, \qquad (104)$$

[19] D. Bohm and J. P. Vigier, Phys. Rev. 109, 1882 (1958).
[20] T. Takabayasi, Progr. Theoret. Phys. (Kyoto), 23, 915 (1960).
[21] J. Weyssenhoff and A. Raabe, Acta Phys. Polon. 9, 8 (1947).
[22] T. Takabayasi, Nuovo Cimento, 13, 532 (1959).
[23] A. Lichnerowicz, Ann. école normale supérieure 60, 247 (1943).

so that

$$\dot{x}_\mu = icb_\mu{}^{(4)}(z). \tag{105}$$

Introducing then the canonical momenta

$$G_\mu = m\dot{x}_\mu + \alpha^{(r)} b_\mu{}^{(r)}(z), \tag{106}$$

and

$$\Pi_{z^\pm} = \partial L/\partial \dot{z}^\pm, \tag{107}$$

we get

$$G_\mu G_\mu = m^2 \dot{x}_\mu \dot{x}_\mu + \alpha^{(r)}\alpha^{(r)} = -M^2c^2 = \text{const}, \tag{108}$$

and (with $H = p\dot{q} - L$):

$$H = \tfrac{1}{2}m\dot{x}_\mu\dot{x}_\mu - \tfrac{1}{2}mc^2 + H_{(i)} \tag{109}$$

$$= (1/2m)(G_\mu G_\mu - \alpha^{(r)}\alpha^{(r)}) - \tfrac{1}{2}mc^2 + H_{(i)} = -mc^2 + H_{(i)}. \tag{110}$$

Now $H_{(i)}$, containing only $z_\mu{}^\pm$, evidently has a vanishing Poisson bracket with H and is a constant of the motion. Relation (109) therefore implies that $\frac{1}{2}m(G_\mu G_\mu - \alpha^{(r)}\alpha^{(r)}) - \frac{1}{2}mc^2$ is also a constant of the motion, so that if we note that it is equal to $-mc^2$ [as a consequence of the squaring of (106)], we see that the rest mass $G_\mu\dot{x}_\mu = -mc^2$ and $\alpha^{(r)}\alpha^{(r)}$ are separate constants of the motion. The existence of the supplementary constant m results from the invariance of L under the one-parameter Abelian group of pure Lorentz transforms along $icb_\mu{}^{(4)}$. All the other constants discovered are evidently related (as they should[15]) to the invariance of L under the translation and G transformations.

The quantization of internal motion is now straightforward. Writing

$$H = -i\hbar\partial/\partial t, \quad G_\mu = -i\hbar\partial_\mu,$$

and

$$\Pi_{z^\pm} = -i\hbar\partial/\partial z^\pm, \tag{111}$$

we get, introducing the total wave field $\Phi(x_\mu, z_\mu{}^\pm, \tau)$ (representing simultaneously the x_μ distribution and the $L-T$ frame orientation),

$$-i\hbar\frac{\partial\phi}{\partial\tau} = \left(-mc^2 + \frac{1}{2I}\{J^{2+} + J^{2-}\}\right)\phi. \tag{112}$$

Relation (108) and this general internal equation (112) yield, if we insert the form

$$\Phi = \exp(-iMc^2\tau/\hbar)\varphi_e(x_\mu)F(z^+, z^-), \tag{113}$$

the two fundamental relations:

$$[\Box - (M^2c^2/\hbar^2)]\varphi_e(x_\mu) = 0 \tag{114a}$$

and

$$(H_{(i)}{}^+ + H_{(i)}{}^- - W)F(z^+, z^-) = 0, \tag{114b}$$

with

$$H_{(i)}{}^\pm = (1/2I)J^{2\pm},$$
$$W = \alpha^{(r)}\alpha^{(r)}/2m$$

being a constant of the motion. Consequently we write

$$(J^{2+} + J^{2-})F(z^+, z^-) = WF(z^+, z^-). \tag{115}$$

Relation (115) is evidently the internal counterpart of the second order Klein-Gordon equation, relation (114a) describing the radial motion of x_μ. As to the symbolical substitution $\Pi_{z^\pm} = -i\hbar\partial/\partial z^\pm$, we know that the correct quantization must be performed by the substitution of commutators to Poisson brackets, but one sees easily that this procedure leads to the same results.

We shall not discuss this model here in more detail.

(B) Now it is clear that the preceding interpretation still contains many unsatisfactory aspects (though it follows step by step the usual presentation of quantum theory) since it leaves open, for example, the problem of the justification of the quantization procedure. In the opinion of two of the authors (D.B. and J.P.V.) a much deeper physical interpretation, which incorporates immediately all results obtained in paper I, can be obtained by starting directly from a field point of view Indeed one can represent the vacuum of subquantum mechanical level by a space-time net of points related by 0-length light rays. Each vertex of such a net has three incoming light rays (equivalent, as Synge[12] has shown, to a four-frame L), and three outgoing light rays in the forward time direction. Since these outgoing rays are also equivalent to a four-frame T, see that such a model can be assumed to be invariant under the external group $L_4(x_\mu)$ defined at each vertex x_μ, which rotates L and T as a block with respect to an arbitrary laboratory frame, and also under our internal group G of relative $L-T$ motions. Since we further know that if the vertex system is dense enough, $L_4(x_\mu)$ is, as Coish[24] has shown, physically equivalent to the usual Lorentz invariance, such a model, if endowed with chaotic "vacuum" or background fluctuations, carries regular quantized excitations which can be classified according to the results of paper I.

The development of this point of view will be published later. Preliminary investigations of one of us[25] show that such a theory can justify the quantization procedure.

(C) Finally the authors, following an idea of Professor Geheniau, have recently discovered that their new isobaric group G can be introduced within the frame of the line of thought initiated by Pauli[26] and Heisenberg,[27] as a possible gauge group for a generalized nonlinear Heisenberg equation. To show this, let us briefly recall Heisenberg's ideas.

Let us start with a wave field χ_{st} which transforms as $\mathfrak{D}(\frac{1}{2},0)$ under SL_4, and as $D(\frac{1}{2})$ under SO_3, and satisfies Heisenberg's equation:

$$\sigma^\nu\partial_\nu\chi = l^2\sigma^\nu\chi(\chi^*\sigma_\nu\chi), \tag{116}$$

[24] H. R. Coish, Phys. Rev. **114**, 383 (1959).
[25] D. Bohm, in *Quantum Theory*, edited by N. R. Bates (Academic Press Inc., New York, 1962), Chap. 6.
[26] W. Pauli, Nuovo Cimento **6**, 204 (1954).
[27] W. Heisenberg, Revs. Mod. Phys. **29**, 269 (1957).

where σ^ν are the four-dimensional Pauli matrices. The introduction of parity in the theory can be obtained by the introduction of a new index $r=1$, 2, namely, the substitution of χ_{str} to χ_{st}, that is χ_{st1} transforming as $\mathfrak{D}(\frac{1}{2},0)$ under SL_4, and χ_{st2} transforming as $\mathfrak{D}(0,\frac{1}{2})$ under SL_4. Relation (116) is then replaced by the system

$$\sigma^\nu\partial_\nu\chi_1=l^2\sigma^\nu\chi_1(\chi_1{}^*\partial_\nu\chi_1), \tag{117a}$$

$$\bar{\sigma}^\nu\partial_\nu\chi_2=l^2\bar{\sigma}^\nu\chi_2(\chi_2{}^*\sigma_\nu\chi_2), \tag{117b}$$

with

$$\bar{\sigma}^k=-\sigma_k,\quad \bar{\sigma}^4=\sigma_4.$$

A further generalization is finally obtained by replacing χ_{str} by $\psi_{s\tau r}$, where τ is an index in isospace which enables ψ to transform like a Dirac spinor in isospace. $\psi_{s\tau r}$ thus transforms according to $D(\frac{1}{2},0)$ or $D(0,\frac{1}{2})$ in isospace and satisfies the relation

$$\tau^\nu\partial_\nu\psi=l^2\tau^\nu\psi(\bar{\psi}\tau_\nu\psi), \tag{118}$$

where $\bar{\psi}=\psi^*\beta$ and τ^ν are the Pauli matrices in isospace. One sees immediately that the new gauge group coincides with G so that by the usual "fusion" procedure one obtains all internal states of our classification.

APPENDIX II

In this paper we have used as an invariant internal isobaric group the four-dimensional complex rotation group $SO_4{}^*$ acting on the complex sphere $S_3{}^*$ taken as the configuration manifold. Moreover we have utilized simultaneously as independent variables the two complex conjugate points z and z^* on this sphere. This raises the following well-known difficulties. First, two conjugated variables are not independent. Secondly, any complex rotation performed on the sphere destroys the complex conjugation between two conjugated points z and z^*. The formalism used in Sec. II is nevertheless correct and can be justified in the following way.[8] Let us consider the direct product of two independent four-dimensional complex spaces $C_4{}^+$ and $C_4{}^-$ and the direct product of two spheres taken in each space, namely $S_3{}^+\times S_3{}^-$. This will be a six-dimensional complex configuration manifold, the fundamental variables being represented by the column

$$Z=\begin{pmatrix} z^+ \\ z^- \end{pmatrix}, \tag{119}$$

composed with two independent complex vectors. We shall now submit this manifold to the transformation

$$Z'=\Omega Z, \tag{120}$$

with

$$\Omega=\begin{pmatrix} \omega & 0 \\ 0 & \omega^* \end{pmatrix}, \tag{121}$$

ω being any elements of $SO_4{}^*$ acting on $C_4{}^+$ and ω^* the complex conjugated transformation which acts on $C_4{}^-$. We now see that the variables z^+ and z^- are truly independent so that we can use partial derivatives with respect to z^+ and z^- and consequently the two independent operators J^+ and J^-. Now we can build without ambiguity the functional space of the functions $F(Z)$ defined on $S_3{}^+\times S_3{}^-$. However, the variables Z with six complex degrees of freedom do not represent a Lorentz transform, unless z^+ and z^- are mutually conjugated. We must therefore consider the intersection of the manifold $S_3{}^+\times S_3{}^-$ with the surface σ defined by

$$z^+=(z^-)^*, \tag{122}$$

and take the traces of the functions $F(z^+,z^-)$ on this intersection, $S_3{}^+\times S_3{}^-/\sigma$, which is now our genuine configuration manifold. This manifold is obviously invariant under the transformations Ω (which build our invariance group) so that the above-mentioned difficulties no longer arise.

This result can be interpreted physically in the following way. The general point Z on the manifold $S_3{}^+\times S_3{}^-$ represents a transformation of the group $SO_4{}^*$ which contains the Lorentz group as the subgroup corresponding to the product of two conjugate complex three-dimensional rotations, that is to the preceding intersection $S_3{}^+\times S_3{}^-/\sigma$. As Einstein and Mayer[28] first pointed out, the transformation from one sphere to the other corresponds to the transition from covariant to contravariant tetrads in real Lorentz space (or $i\rightarrow -i$ transformations in Minkowski space) as they are related, respectively, to self-dual and antidual skew tensors. Independent motions on both spheres would amount to independent variations of covariant and contravariant quantities, something that never happens in real physical space.

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[28] A. Einstein and M. Mayer, Sitzber Preuss. Akad. Wiss. Physik.-Math. Kl. 522 (1932).